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Abstract. Moduli spaces of semistable sheaves on a K3 or abelian surface with respect to a general ample divisor are shown to be locally factorial, with the exception of symmetric products of a K3 or abelian surface and the class of moduli spaces found by O'Grady. Consequently, since singular moduli space that do not belong to these exceptional cases have singularities in codimension ≥ 4 they do no admit projective symplectic resolutions.

1. Introduction

How to construct irreducible holomorphic symplectic manifolds? Except for the variety of Beauville and Donagi [4] all known examples arise from moduli spaces of semistable sheaves on a K3 or abelian surface.

For every element v in the Mukai lattice $H^{\mathrm{even}}(X,\mathbb{Z})$ of a polarised K3 or abelian surface (X,H) there is an associated moduli space M_v that parametrises polystable sheaves E with Mukai vector $v=v(E):=ch(E)\sqrt{\mathrm{td}(X)}$. If H and v are chosen to the effect that no strictly semistable sheaves exist, i.e. every semistable sheaf is automatically stable, then M_v is a projective holomorphically symplectic manifold due to Mukai [21].

In the opposite case, M_v is singular and one may ask whether M_v at least admits a projective symplectic resolution. This question has been raised and successfully answered in two cases by O'Grady [23, 24], leading to two new deformation classes of irreducible holomorphic symplectic manifolds.

In this paper we give a complete answer to O'Grady's question for general ample divisors H and moduli spaces whose expected dimension $2+\langle v,v\rangle$ is ≥ 4. The answer depends essentially only on the divisibility of the Mukai vector $v\in H^{\mathrm{even}}(X,\mathbb{Z})$ and the dimension of the moduli space. We may write $v=mv_0$ with a primitive Mukai vector $v_0=(r,c,a)$ and a multiplicity $m\in\mathbb{N}$. Suppose for simplicity that $r>0$, and let H denote a v–general ample divisor. Then every semistable sheaf E with Mukai vector $v(E)=v_0$ is stable, and a necessary and sufficient condition for the existence of E is that $c\in\mathrm{NS}(X)$ and that $\langle v_0,v_0\rangle\geq -2$. There are five principal cases to distinguish:

1) If $\langle v_0,v_0\rangle=-2$, then Mukai has shown that M_{v_0} consists of a single point $[E_0]$ only. As the expected dimension of the moduli space M_{mv_0} is negative for

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$m > 0$ there are no stable sheaves in this case, and it follows by induction that any semistable sheaf must be of the form $E_0^{\oplus m}$. Hence M_v is a single point as well.

2) If $\langle v_0, v_0\rangle = 0$, the moduli space M_{v_0} is again a K3 surface or an abelian surface if X is K3 or abelian due to beautiful results of Mukai. It turns out that any semistable sheaf E with $v(E) = mv_0$ is S-equivalent to a direct sum $E = E_1 \oplus \ldots \oplus E_m$ with stable sheaves $[E_i] \in M_{v_0}$. It follows that $M_v = S^m(M_{v_0})$. Thus the moduli spaces are singular in codimension 2, but admit symplectic resolutions in terms of the Hilbert scheme $\mathrm{Hilb}^m(M_{v_0}) \to M_v$.

3) Assume now that $\langle v_0, v_0\rangle \geq 2$. Due to the combined efforts of many authors, with important steps taken by Mukai, Huybrechts, O'Grady and Yoshioka, one finally has the following result [27]: M_{v_0} is a smooth symplectic variety that is deformation equivalent to $\mathrm{Hilb}^{1+\frac{1}{2}\langle v_0,v_0\rangle}(X)$, if X is a K3-surface, and to $\mathrm{Pic}_0(X) \times \mathrm{Hilb}^{\frac{1}{2}\langle v_0,v_0\rangle}$, if X is an abelian surface.

Assume in addition that $m \geq 2$. The main result of this article implies that one has to further distinguish the following two cases:

4) Let $\langle v_0, v_0\rangle = 2$ and $m = 2$. The moduli spaces $M_{K3}(2; 0, 4)$ and $M_{Ab}(2; 0, 2)$ studied by O'Grady [23, 24] and Rapagnetta [25] fall into this class. The moduli space M_v has dimension 10, its singular locus has codimension 2 and is in fact isomorphic to $S^2 M_{v_0}$. As shown in [18], the symplectic desingularisations constructed by O'Grady exist for all Mukai vectors in this class and can be obtained by blowing-up the reduced singular locus.

5) In all other cases our main result states:

Theorem A — *If either $m \geq 2$ and $\langle v_0, v_0\rangle > 2$ or $m > 2$ and $\langle v_0, v_0\rangle \geq 2$, then M_{mv_0} is a locally factorial singular symplectic variety.*

As an immediate application one obtains:

Theorem B — *Under the hypotheses of Theorem A, M_{mv_0} does not admit a proper symplectic resolution.*

Under some technical hypotheses theorems A and B hold as well for semistable torsion sheaves (see the main text). Partial results for Theorem B in the case $m = 2$ have been obtained previously by two of us [17], and, independently and with different methods, by Kiem and Choy [15, 16].

We note that our approach is rather general; our main technical result, Proposition 3.5, is essentially a linear-algebraic fact. Therefore, we expect that results similar to Theorem A and B might hold in other situations with similar geometry – in particular, for the moduli spaces of flat connections on an algebraic curve. In fact, Proposition 3.5 is a statement about quiver varieties of H. Nakajima [22], although the numerical data corresponding to our quivers are specifically excluded from consideration in [22] (which is not surprising as one of the results of [22] is that any quiver variety considered there *does* admit a symplectic resolution). Thus our approach and our Proposition 3.5 might be used wherever one finds quiver varieties of the same type.

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Many authors have worked on moduli of sheaves on K3 and abelian surfaces ever since the seminal work of Mukai. The most general results for our purposes have been obtained by Yoshioka [27]. We refer to the textbook [13] and Yoshioka's paper for further references and general information on semistable sheaves and their moduli spaces.

2. Notation and conventions, plan of the paper

2.1. **The underlying surface.** Throughout this article X will denote a complex projective K3 or abelian surface with a fixed symplectic structure, i.e. an isomorphism $H^2(X, \mathcal{O}_X) \cong \mathbb{C}$, and a fixed ample divisor H.

The even integral cohomology $H^{\mathrm{even}}(X, \mathbb{Z})$ is equipped with a pairing

$$\langle v, w \rangle := -\int_X v w^\vee,$$

where $w^\vee = (-1)^i w$ for $w \in H^{2i}(X, \mathbb{Z})$. Following Mukai we associate to each coherent sheaf E its Mukai vector

$$v(E) := \mathrm{ch}(E)\sqrt{\mathrm{td}(X)} \in H^{\mathrm{even}}(X, \mathbb{Z}).$$

The Hilbert polynomial of E with respect to an ample divisor H can be expressed in terms of its Mukai vector as follows:

$$\chi(E \otimes \mathcal{O}_X(mH)) = -\langle v(E), v(\mathcal{O}_X(-mH))\rangle =: P_v(m).$$

2.2. **Semistable sheaves.** Stability or semistability of a coherent sheaf is defined with respect to a fixed ample divisor H. We let M_v denote the moduli space of semistable sheaves with Mukai vector v. Closed points of M_v are in natural bijection with polystable sheaves E. Points corresponding to stable sheaves form a – possibly empty – open subset $M_v^s \subset M_v$.

Semistable sheaves may have two–, one– or zero–dimensional support. Stability in the first case was defined by Maruyama and Gieseker, the generalisation to pure sheaves of arbitrary dimension is due to Simpson. In the rest of the paper we exclude once for all the case of zero-dimensional sheaves as being well-known: if the Mukai vector is $v = (0, 0, a)$ then $M_v \cong S^a X$, the symmetric product of X, and the Hilbert-Chow morphism $\mathrm{Hilb}^a(X) \to M_v$ provides a symplectic resolution.

2.3. **General assumptions.** An element $v_0 \in H^{\text{even}}(X,\mathbb{Z})$ is primitive if it is not an integral multiple of another lattice element. Given a non-trivial element $v \in H^{\text{even}}(X,\mathbb{Z})$ we may always decompose it as $v = mv_0$ with a primitive element v_0 and a multiplicity $m \in \mathbb{N}$. Throughout this article we assume that $v_0 = (r_0, c_0, a_0)$ has the following properties:

$$(*) \qquad \begin{cases} \text{Either } r_0 > 0 \text{ and } c_0 \in \mathrm{NS}(X), \\ \qquad \text{or } r_0 = 0,\ c_0 \in \mathrm{NS}(X) \text{ is effective, and } a_0 \neq 0; \\ \langle v_0, v_0 \rangle \geq 2. \end{cases}$$

The results of this paper suggest to distinguish systematically between the following three cases for a Mukai vector v satisfying assumptions $(*)$:

(A) $m = 1$.
(B) $m = 2$ and $\langle v_0, v_0 \rangle = 2$.
(C) $m \geq 3$, or $m = 2$ and $\langle v_0, v_0 \rangle \geq 4$.

2.4. **General ample divisors.** The significance of $(*)$ lies in the fact that one has the notion of a v–general ample divisor H: there is a systems of hyperplanes in the ample cone of X, called v–walls, that is countable but locally finite for torsion free sheaves ([13], ch. 4C) and finite for torsion sheaves ([27], sec. 1.4.) with the following property: if H is v–general, i.e. if H is not contained in any v–wall, then for every direct summand E' of a polystable sheaf E with $v(E) = v$ one has $v(E') \in \mathbb{Q}v(E)$.

Let H be a v_0–general ample divisor and consider the following assertions:

$$(**) \quad M_{v_0} \text{ is non-empty.}$$
$$(***) \quad M_{v_0} \text{ is irreducible.}$$

Yoshioka shows in [27], Thm 0.1 and Thm 8.1, that $(*)$ implies $(**)$ and $(***)$ except when X is a K3 surface, $r_0 = 0$ and c_0 is not ample. Moreover he has communicated to us an unpublished note that fills this gap, so that $(**)$ and $(***)$ are consequences of $(*)$. An essential technique in Yoshioka's work is the deformation of the underlying surface; the arguments are rather involved. For the irreducibility part $(***)$ we give a new and direct proof, based on an old and beautiful idea of Mukai, see Theorem 4.1.

2.5. **Elements of the construction of moduli spaces.** We need to recall some basic elements of the construction machinery of moduli spaces of sheaves following the approach of Simpson [26] (see also [13], ch. 4). Let v be a Mukai vector satisfying $(*)$ and let P_v denote the corresponding Hilbert polynomial. Choose a sufficiently large integer $k = k(v)$ and put $N = P_v(k)$, $\mathcal{H} := \mathcal{O}_X(-kH)^{\oplus N}$. Then there is a closed subscheme $R \subset \mathrm{Quot}_{X,H}(\mathcal{H}, P)$ with the following property: a closed point

$$[q : \mathcal{H} \to E] \in R$$

is stable or semistable with respect to the canonical $\mathrm{PGl}(N)$–action and the corresponding linearisation of the determinant line bundle on R if and only if q induces an isomorphism $\mathbb{C}^N \to H^0(X, E(kH))$ and if E is stable or semistable, respectively. Let $R^s \subset R^{ss} \subset R$ denote the open subsets of stable and semistable points, respectively. Then

$$R^{ss} /\!\!/ \mathrm{PGl}(N)) \cong M_v \qquad \text{and} \qquad R^s /\!\!/ \mathrm{PGl}(N) \cong M_v^s.$$

Let $\pi : R^{ss} \to M_v$ denote the quotient map. The orbit of a point $[q : \mathcal{H} \to E]$ is closed in R^{ss} if and only if E is polystable. In that case, the stabiliser subgroup of $[q]$ in $\mathrm{PGl}(N)$ is canonically isomorphic to $\mathrm{PAut}(E) = \mathrm{Aut}(E)/\mathbb{C}^*$. Moreover, by Luna's slice theorem there is a $\mathrm{PAut}(E)$–invariant subscheme $S \subset R^{ss}$, containing $[q]$, such that the canonical morphisms

$$(\mathrm{PGl}(N) \times S) /\!\!/ \mathrm{PAut}(E) \to R^{ss} \qquad \text{and} \qquad S /\!\!/ \mathrm{PAut}(E) \to M$$

are étale. The Zariski tangent space $T_{[q]}S$ is isomorphic to $\mathrm{Ext}^1(E, E)$.

2.6. **Local description.** The completion of the local ring $\mathcal{O}_{S,[q]}$ has the following deformation theoretic description:

Let $\mathbb{C}[\mathrm{Ext}^1(E, E)]$ denote the ring of polynomial functions on $\mathrm{Ext}^1(E, E)$ and let $A := \mathbb{C}[\mathrm{Ext}^1(E, E)]^\wedge$ denote its completion at the maximal ideal $\mathfrak{M}$ of functions vanishing at 0. There is a trace map $\mathrm{tr} : \mathrm{Ext}^2(E, E) \to H^2(\mathcal{O}_X)$. We denote its kernel by $\mathrm{Ext}^2(E, E)_0$. The automorphism group $\mathrm{Aut}(E)$ naturally acts on $\mathrm{Ext}^1(E, E)$ and $\mathrm{Ext}^2(E, E)_0$ by conjugation. Since the scalar multiples of the identity act trivially we actually have an action of the projective automorphism group $\mathrm{PAut}(E) = \mathrm{Aut}(E)/\mathbb{C}^*$. There is a linear map

$$\kappa : \mathrm{Ext}^2(E, E)_0^* \longrightarrow \mathbb{C}[\mathrm{Ext}^1(E, E)]^\wedge,$$

the so-called Kuranishi map, with the following properties:

1. κ is $\mathrm{PAut}(E)$–equivariant.
2. Let I be the ideal generated by the image of κ. Then there are isomorphisms of complete rings
$$\widehat{\mathcal{O}}_{S,[q]} \cong A/I \quad \text{and} \quad \widehat{\mathcal{O}}_{M_v,[E]} \cong (A/I)^{\mathrm{PAut}(E)}.$$
3. For every linear form $\varphi \in \mathrm{Ext}^2(E, E)_0^*$ one has, for $e \in \mathrm{Ext}^1(E, E)$,
$$\kappa(\varphi)(e) = \tfrac{1}{2}\varphi(e \cup e) + \text{higher order terms in } e.$$

2.7. **Passage to the normal cone.** Let $J \subset \mathbb{C}[\mathrm{Ext}^1(E, E)]$ denote the ideal generated by the image of the quadratic part of κ:

$$\kappa_2 : \mathrm{Ext}^2(E, E)_0^* \longrightarrow S^2\,\mathrm{Ext}^1(E, E)^*, \quad \varphi \mapsto (e \mapsto \tfrac{1}{2}\varphi(e \cup e))$$

Then J is the ideal of the null-fibre $F = \mu^{-1}(0)$ of the morphism

$$\mu : \mathrm{Ext}^1(E, E) \longrightarrow \mathrm{Ext}^2(E, E)_0, \quad \mu(e) = \tfrac{1}{2}(e \cup e).$$

The ideals $I \subset \mathbb{C}[\mathrm{Ext}^1(E, E)]^\wedge$ and $J \subset \mathbb{C}[\mathrm{Ext}^1(E, E)]$ are related as follows. The graded ring $\mathrm{gr}\, A$ associated to the $\mathfrak{m}$-adic filtration on $A = \mathbb{C}[\mathrm{Ext}^1(E, E)]^\wedge$ is canonically isomorphic to $\mathbb{C}[\mathrm{Ext}^1(E, E)]$. For any ideal $\mathfrak{a} \subset A$ let $\mathrm{in}(\mathfrak{a}) \subset \mathrm{gr}\, A$ denote the ideal generated by the leading terms $\mathrm{in}(f)$ of all elements $f \in \mathfrak{a}$. Then property 3 of the Kuranishi–map says that

$$J \subset \mathrm{in}(I).$$

Hence there is the following chain of inequalities:

$$\begin{aligned} \dim(F) &= \dim\,(\mathrm{gr}\,A)/J \\ &\geq \dim\,(\mathrm{gr}\,A)/\,\mathrm{in}(I) \;=\; \dim\,\mathrm{gr}(A/I) \;=\; \dim(A/I) \\ &\geq \dim \mathrm{Ext}^1(E,E) - \dim \mathrm{Ext}^2(E,E)_0, \end{aligned} \tag{2.1}$$

where the last inequality comes from the fact that A is regular of dimension $= \dim \mathrm{Ext}^1(E,E)$ and I is generated by $\dim \mathrm{Ext}^2(E,E)_0$ elements.

We need to describe μ in greater detail; the resulting description is similar to Nakajima's construction of the so-called quiver varieties [22]. Write

$$E = \bigoplus_{i=1}^{s} W_i \otimes E_i \tag{2.2}$$

with pairwise non-isomorphic stable sheaves E_i and vector spaces W_i of dimension n_i. Let $W_{ij} := \mathrm{Hom}(W_i, W_j)$ and $V_{ij} := \mathrm{Ext}^1(E_i, E_j)$. Then

$$\mathrm{End}(E) = \bigoplus_i W_{ii}, \quad \mathrm{Ext}^1(E,E) = \bigoplus_{i,j} W_{ij} \otimes V_{ij}, \quad \mathrm{Ext}^2(E,E) = \bigoplus_i W_{ii}.$$

The automorphism group

$$\mathrm{Aut}(E) = \prod_i \mathrm{Aut}(W_i) \cong \prod_i \mathrm{Gl}(n_i) =: G(n)$$

acts on $\mathrm{Ext}^1(E,E)$ by conjugation on the first factor in each direct summand. By Serre-Duality, the pairing

$$V_{ij} \otimes V_{ji} \to \mathbb{C}, \quad e \otimes e' \mapsto \mathrm{tr}(e' \cup e)$$

is non-degenerate and antisymmetric. This yields a symplectic form ω on $\mathrm{Ext}^1(E,E)$ such that $W_{ij} \otimes V_{ij}$ and $W_{ab} \otimes V_{ab}$ are perpendicular, unless $i = b$ and $j = a$, in which case

$$\omega : (W_{ij} \otimes V_{ij}) \otimes (W_{ji} \otimes V_{ji}) \longrightarrow \mathbb{C}, \quad \omega(A \otimes e, A' \otimes e') = \mathrm{tr}(A'A)\,\mathrm{tr}(e' \cup e).$$

Moreover, the quadratic map $\mu : \mathrm{Ext}^1(E,E) \to \mathrm{Ext}^2(E,E)_0$ is given by

$$\mu\left(\sum_{ij}\sum_k A_{ij}^k \otimes e_{ij}^k\right) = \sum_{ij}\sum_{k,\ell} A_{ij}^k A_{ji}^\ell\,\mathrm{tr}(e_{ij}^k e_{ji}^\ell). \tag{2.3}$$

2.8. **Strategy.** In general we do not know how to compute the Kuranishi map explicitly. However, the explicit description of the quadratic part μ given above allows for a detailed study of the fibre $F := \mu^{-1}(0) \subset \mathrm{Ext}^1(E,E)$. The passage from κ to μ corresponds to the passage from the local ring $\widehat{\mathcal{O}}_{S,[q]}$ to the coordinate ring $\mathcal{O}_F$ of its tangent cone.

In section 3 we show that under certain hypotheses the fibre F is an irreducible normal complete intersection which is, in case (C), regular in codimension ≤ 3 and state consequences for the local rings $\mathcal{O}_{R^{ss},[q]}$ of points $[q]$ in closed orbits of R^{ss}.

Section 4 contains a basic irreducibility result for moduli spaces of sheaves on a K3 or abelian surface.

In section 5 it is proved — under the hypothesis that the ample divisor is v–general — that the moduli space M is a non-empty irreducible normal variety of expected dimension, and that it is locally factorial in case (C). As an application we show in section 6 that in case (C) the moduli space does not admit a symplectic resolution.

3. Symplectic reduction

3.1. **The symplectic momentum map.** Let U be a smooth affine algebraic variety over $\mathbb{C}$ endowed with a symplectic form ω. Let G be a reductive group that acts on U preserving ω. This action induces an infinitesimal action of the Lie algebra $\mathfrak{g}$ of G, i.e. a homomorphism of Lie algebras $\mathfrak{g} \to \Gamma(U, T_U)$. We denote the vector field corresponding to $A \in \mathfrak{g}$ at $x \in U$ by A_x. A momentum map for the action is a G-equivariant morphism $\mu : U \to \mathfrak{g}^*$ with the property that $d\mu_x(\xi)(A) = \omega(\xi, A_x)$ for all $x \in U$ and $\xi \in T_xU$. If a momentum map exists, it is unique up to an additive constant in $(\mathfrak{g}^*)^G$.

Let $\mu : U \to \mathfrak{g}^*$ be a momentum map with null-fibre $F := \mu^{-1}(0)$.

Lemma 3.2. — *Let $x \in F$ be a point with stabiliser subgroup $H \subset G$. Then the image of $d\mu_x : T_xU \to \mathfrak{g}^*$ is $(\mathfrak{g}/\mathfrak{h})^* = \mathfrak{h}^\perp$, where $\mathfrak{h} \subset \mathfrak{g}$ denotes the Lie algebra of H. In particular, if H is finite then $d\mu_x$ has maximal rank and F is regular at x of dimension* $\dim(U) - \dim(G)$.

Proof. The image $d\mu_x$ annihilates $A \in \mathfrak{g}$ if and only if $\omega(\xi, A_x) = 0$ for all $\xi \in T_xU$, i.e. if A_x is perpendicular to T_xU with respect to ω. As ω is non-degenerate, this is equivalent to saying that A_x vanishes, hence is a tangent vector to the stabiliser subgroup H. □

Lemma 3.3. — *Let $\mu : U \to \mathfrak{g}^*$ be a momentum map with null-fibre F. Let $Z \subset F$ be the closed subset of points with non-finite stabiliser group. Let $d = \dim U - \dim \mathfrak{g}$.*

1. *If $\dim(Z) \leq d - 1$, then F is a reduced complete intersection of dimension d.*
2. *If $\dim(Z) \leq d - 2$, then F is normal.*

Proof. Every irreducible component of F must have dimension $\geq d$ since F is cut out by $\dim \mathfrak{g}$ equations. By Lemma 3.2, F has dimension d in each point $x \in F \setminus Z$. If $\dim Z < d$, then $F \setminus Z$ is dense in F and every irreducible component has precisely dimension d. Hence F is a complete intersection and in particular Cohen-Macaulay ([2], Cor. III 4.5). Since $F \setminus Z$ is smooth, F in addition satisfies condition R_0 and is therefore reduced ([2], Prop. VII 2.2). If in addition $\dim(Z) \leq d - 2$, then F is regular in codimension 1 and normal by Serre's criterion ([2], Cor. VII 2.13). □

3.4. **The key estimate.** We want to apply the lemma to the following particular situation, that arises in the study of local rings of the moduli space of sheaves.

Set-up: Let $W_1, \ldots, W_s$ be a sequence of vector spaces, $s \geq 1$. The dimensions $n_i = \dim(W_i)$ form the components of a vector $n \in \mathbb{N}_0^s$. Furthermore, let $W_{ij} = \mathrm{Hom}(W_i, W_j)$. There is a natural symmetric pairing

$$W_{ij} \otimes W_{ji} \to \mathbb{C}, \qquad (A, B) \mapsto \mathrm{tr}(BA).$$

Moreover, let V_{ij}, $1 \leq i, j \leq s$, be vector spaces, equipped with non-degenerate pairings

$$\omega_{ij} : V_{ij} \otimes V_{ji} \to \mathbb{C},$$

that are skew-symmetric in the sense that $\omega_{ij}(e, e') = -\omega_{ji}(e', e)$. Then the vector space $U(n) := \bigoplus_{i,j} W_{ij} \otimes V_{ij}$ carries a natural symplectic form ω with the property that $W_{ij} \otimes V_{ij}$ is perpendicular to all $W_{ab} \otimes V_{ab}$, $(a, b) \neq (j, i)$ and

$$\omega : \Big(W_{ij} \otimes V_{ij}\Big) \otimes \Big(W_{ji} \otimes V_{ji}\Big) \longrightarrow \mathbb{C}, \quad (A \otimes e) \otimes (A' \otimes e') \mapsto \mathrm{tr}(A'A)\omega_{ij}(e, e').$$

In the following arguments the vector spaces V_{ij} are fixed and chosen once for all, whereas the sequence of vector spaces W_i can be replaced by appropriate subspaces etc. We will argue by induction over the dimension vector n as an element in the monoid $\mathbb{N}_0^s$. Most objects defined below will therefore be indexed by n, like the space $U(n)$ above, even if this is not quite accurate as they really depend on the spaces W_i.

The group $G(n) = \prod_i \mathrm{Aut}(W_i)$ acts on $U(n)$ by conjugation on the first factors in the decomposition. The subgroup of scalars $\mathbb{C}^* \subset G(n)$ acts trivially. Let $PG(n) := G(n)/\mathbb{C}^*$. The action of $PG(n)$ on $U(n)$ preserves the symplectic structure. The moment map for the action is

$$\begin{array}{rcl} \mu(n) : U(n) & \longrightarrow & \mathfrak{pg}(n)^* \simeq \mathrm{Ker}\Big(\bigoplus_i \mathfrak{gl}(n_i) \xrightarrow{\mathrm{tr}} \mathbb{C}\Big), \\ \sum_{i,j,k} A_{ij}^{(k)} v_{ij}^{(k)} & \mapsto & \sum_{k,\ell} \sum_{i,j} A_{ij}^{(k)} A_{ji}^{(\ell)} \,\mathrm{tr}(v_{ij}^{(k)} \cup v_{ji}^{(\ell)}) \end{array}$$

Let $F(n) := \mu(n)^{-1}(0) \subset U(n)$ denote the null-fibre of the moment map. The structure of $F(n)$ depends only on n and the dimensions $d_{ij} := \dim(V_{ij})$. Let D denote the matrix (d_{ij}) and let $a := \min\{d_{ij} - 2\delta_{ij}\}$.

Proposition 3.5. — *Assume that $a \geq 2$. Then $F(n)$ is an irreducible normal complete intersection of dimension $d := n^t(D - I)n + 1$. Moreover, $F(n)$ is regular in codimension ≤ 3 with the possible exception of the two cases*

1. *$n = (1, 1)$, $d_{12} = 2$, and*
2. *$n = (2)$, $d_{11} = 4$.*

Proof. 1. Since $\dim(U(n)) = \sum_{i,j} n_i n_j d_{ij}$ and since the range of μ has dimension $\sum_i n_i n_i - 1$, the expected dimension of $F(n)$ is

$$d = \sum_{i,j} n_i n_j d_{ij} - \sum_i n_i n_i + 1 = n^t(D - I)n + 1.$$

Also, $F(n)$ is a cone and hence connected. By Lemma 3.3, it suffices to show that the locus Z of points in $F(n)$ with non-trivial stabiliser in $PG(n)$ has dimension

$\leq d-4$ in general and $\leq d-3$ in the two exceptional cases. This will be done by induction on the dimension vector $n \in \mathbb{N}_0^s$.

The induction starts with $n = (0, \ldots, 1, \ldots, 0)$, in which case the statement is trivial. So let $n \in \mathbb{N}_0^s$ be an arbitrary element and assume that the proposition holds for all $n' \in \mathbb{N}_0^s$ such that $0 < \sum_i n'_i < \sum_i n_i$.

2. We can analyse Z as follows: Let $g \in G(n)$, $g \notin \mathbb{C}^*$, and consider the corresponding fixed point locus $F(n)^g$. The image $G(n)F(n)^g$ of the morphism $\varphi : G(n) \times F(n)^g \to F(n)$, $(g', x) \mapsto g'x$, consists of all points $y \in F$ whose stabiliser subgroup $G(n)_y$ contains an element conjugate to g. Suppose that $H \subset G(n)$ is a subgroup that stabilises the fixed point set $F(n)^g$. Then we can bound the dimension of the fibres of φ by $\dim(H)$. It follows that $\dim(G(n)F(n)^g) \leq \dim F(n)^g + \dim G(n) - \dim H$. In the following we will describe a finite set of elements g such that Z is covered by the corresponding sets $G(n)F(n)^g$ and such that for each g one has $\dim(G(n)F(n)^g) \leq d-3$ or $\leq d-4$. This gives the desired bound for $\dim(Z)$.

3. Let $g = (g_1, \ldots, g_s) \in G(n)$, $g \notin \mathbb{C}^*$. We distinguish three cases:

3.1. *Case: g is semisimple.* For each $\lambda \in \mathbb{C}$ consider the eigenspaces $W_i(\lambda) \subset W_i$ of g_i, and let $n_i(\lambda) = \dim(W_i(\lambda))$, $n(\lambda) = (n_i(\lambda))_i$. Then $n = \sum_\lambda n(\lambda)$. There is a decomposition

$$U(n)^g = \bigoplus_\lambda U(n(\lambda)), \quad U(n(\lambda)) = \bigoplus_{ij} \mathrm{Hom}(W_i(\lambda), W_j(\lambda)) \otimes V_{ij}.$$

Moreover, the restriction of the momentum map to the fixed point locus splits into a product of momentum maps for each $U(n(\lambda))$:

$$\mu(n)|_{U(n)^g} = \prod_\lambda \mu(n(\lambda)), \quad \mu(n(\lambda)) : U(n(\lambda)) \longrightarrow \mathfrak{pg}(n(\lambda))^*.$$

It follows that

$$F(n)^g = \prod_\lambda F(n(\lambda)) \quad \text{with } F(n(\lambda)) = \mu(n(\lambda))^{-1}(0).$$

By induction, we have

$$\dim(F(n)^g) = \sum_\lambda \dim(F(n(\lambda)) = \sum_\lambda{}' \Big(n(\lambda)^t (D-I) n(\lambda) + 1 \Big),$$

where $\sum'$ indicates that we only sum over all λ with $n(\lambda) \neq 0$.

Next, $U(n)^g$ is stabilised by $H = \prod_\lambda G(n(\lambda))$, a subgroup in $G(n)$ of codimension $n^t n - \sum_\lambda n(\lambda)^t n(\lambda)$. We obtain the following upper bound for the dimension of $G(n)F(n)^g$:

$$\dim(G(n)F(n)^g) \leq \sum_\lambda{}' \Big(n(\lambda)^t (D-2I) n(\lambda) + 1 \Big) + n^t n.$$

Note that $\nu := |\{\lambda \mid n(\lambda) \neq 0\}| \geq 2$, since $g \notin \mathbb{C}^*$. The difference of $\dim(G(n)F(n)^g)$ to the expected dimension of $F(n)$ is therefore bounded below by

$$\begin{aligned}\Delta &:= \big(n^t(D-2I)n+1\big) - \sum_\lambda{}' \big(n(\lambda)^t(D-2I)n(\lambda)+1\big)\\ &= \sum_{\lambda\neq\mu} n(\lambda)^t(D-2I)n(\mu) - (\nu-1)\\ &\geq 2\nu(\nu-1)-(\nu-1) = (2\nu-1)(\nu-1) \geq 3.\end{aligned}$$

Clearly, $\Delta \geq 4$ for $\nu \geq 3$. Assume that $\nu = 2$, say with the distinct eigenvalues λ and λ'. Then

$$\Delta \geq 2\sum_{i,j} n_i(\lambda')n_j(\lambda'')(d_{ij}-2\delta_{ij}) - 1 \geq 2a\sum_i n_i(\lambda')\sum_i n_i(\lambda) - 1.$$

Thus $\Delta = 3$ implies $a = 2$ and $\sum_i n_i(\lambda) = 1 = \sum_i n_i(\lambda')$. Hence there are only the following exceptional cases:

1. $s = 1$, $n = 2$, $d_{11} = 2 + 2\delta_{11} = 4$, or
2. $s = 2$, $n = (1,1)$ and $d_{12} = d_{21} = 2$.

If a point $f \in F(n)$ is fixed by a semisimple element, it is also fixed by a whole subtorus $T \subset G(n)$. Up to a conjugation, there is only a finite number of such subtori $T_i \subset G(n)$. Choosing an element $g_i \in T_i$ in each of these subtori, we see that the union of all sets $G(n)F(n)^g$, g semisimple, is covered by the finite union of all sets $G(n)F(n)^{g_i}$.

3.2. *Case: g is unipotent.* We may write $g = 1 + h$, with a non-zero nilpotent element $h = (h_1, \ldots, h_s) \in \bigoplus_i \mathrm{End}(W_i)$. Let $K_i^{(\ell)} := \ker(h_i^\ell) \subset W_i$ and $m_i(\ell) := \dim K_i^{(\ell)}$ for all $\ell \in \mathbb{N}_0$. There is a filtration

$$0 = K_i^{(0)} \subset K_i^{(1)} \subset \ldots = W_i.$$

For each level $\ell > 0$ we choose a graded complement $W_i^{(\ell)}$ to $hK_i^{(\ell+1)} + K_i^{(\ell-1)}$ in $K^{(\ell)}$ and let $n_i^{(\ell)} = \dim W_i^{(\ell)}$. (We note that this is an instance of the so-called Jacobson-Morozov-Deligne filtration associated to a nilpotent element, see [8, 1.6]; the spaces $W_i^{(\ell)}$ are the primitive subspaces with respect to an sl_2-triple containing h.)

Suppose that $A = (A_{ij}) \in \bigoplus_{ij} \mathrm{Hom}(W_i, W_j)$ commutes with h. Then A_{ij} is completely determined by its value on the spaces $W_i^{(\ell)}$, $\ell \in \mathbb{N}$, and conversely, any value of $A_{ij} : W_i^{(\ell)} \to K_j^{(\ell)}$ can be prescribed. The composition with the canonical projection $K_j^{(\ell)} \to W_j^{(\ell)}$ defines a homomorphism $A_{ij}^{(\ell)} : W_i^{(\ell)} \longrightarrow W_j^{(\ell)}$, and the map

$$\Phi : \left(\bigoplus_{ij} \mathrm{Hom}(W_i, W_j)\right)^g \longrightarrow \bigoplus_\ell \left(\bigoplus_{ij} \mathrm{Hom}(W_i^{(\ell)}, W_j^{(\ell)})\right), \quad (A_{ij}) \mapsto (A_{ij}^{(\ell)}),$$

is a ring homomorphism. Let

$$\Phi_V : U(n)^g = \left(\bigoplus_{i,j} W_{ij} \otimes V_{ij}\right)^g \longrightarrow \bigoplus_\ell U(n^{(\ell)}) = \bigoplus_\ell \bigoplus_{i,j} \mathrm{Hom}(W_i^{(\ell)}, W_j^{(\ell)}) \otimes V_{ij}$$

be analogously defined. Then $\Phi_V(F(n)^g)\subset\prod_\ell F(n^{(\ell)})$, and the fibres of Φ_V have dimension $\sum_\ell n^{(\ell)t}D(m^{(\ell)}-n^{(\ell)})$. By induction, this yields the bound

$$\begin{aligned}\dim(F(n)^g) &\leq \sum_\ell \dim(F(n(\ell))+\dim(\ker(\Phi_V))\\ &= \sum_\ell{}' \big(n^{(\ell)t}(D-I)n^{(\ell)}+1\big)+\sum_\ell n^{(\ell)t}D(m^{(\ell)}-n^{(\ell)}),\end{aligned}$$

where $\sum'$ signifies summation over all ℓ with $n^{(\ell)}\neq 0$. Moreover, the centraliser $H\subset G(n)$ of g is an open subset in

$$\left(\bigoplus_i \operatorname{End}(W_i)\right)^g\cong\bigoplus_\ell\bigoplus_i \operatorname{Hom}(W_i^{(\ell)},K_i^{(\ell)})$$

and therefore has dimension $\dim(H)=\sum_\ell n^{(\ell)t}m^{(\ell)}$. Connecting these pieces of information we obtain

$$\begin{aligned}\dim(G(n)F(n)^g) &\leq \dim(F(n)^g)+\dim(G(n))-\dim(H)\\ &\leq \sum_\ell{}' \big(n^{(\ell)t}(D-I)n^{(\ell)}+1\big)+\sum_\ell n^{(\ell)t}D(m^{(\ell)}-n^{(\ell)})\\ &\quad +n^tn-\sum_\ell n^{(\ell)t}m^{(\ell)}.\end{aligned}$$

The difference of the last expression to the expected dimension of $F(n)$ is

$$\Delta:=\Big[n^t(D-I)n-n^tn+1\Big]-\sum_\ell{}'\Big[n^{(\ell)t}(D-I)m^{(\ell)}-n^{(\ell)t}n^{(\ell)}+1\Big].$$

Note that the two bracketed expressions are not quite symmetric to each other due to the presence of $m^{(\ell)}$ instead on $n^{(\ell)}$. We can get rid of n and $m^{(\ell)}$ due to the relations

$$m^{(\ell)}=\sum_k n^{(k)}\min\{k,\ell\},\quad n=\sum_k n^{(k)}k,$$

and can rewrite the bound Δ in terms of the $n^{(k)}$ as follows:

$$\Delta=\sum_{\ell,k}n^{(k)t}(D-I)n^{(\ell)}\big(k\ell-\min\{k,\ell\}\big)-\sum_{k,\ell}n^{(k)t}n^{(\ell)}k\ell+\sum_k{}'\Big(n^{(k)t}n^{(k)}-1\Big)+1$$

Reorganise the sum in collecting those terms that contain $n^{(1)}$:

$$\begin{aligned}\Delta=1+\Big[-1+2n^{(1)t}\sum_{k\geq2}\big((k-1)(D-2I)-I\big)n^{(k)}\Big]+\sum_{k\geq 2}{}'\big(n^{(k)t}n^{(k)}-1\big)\\ +\sum_{k,\ell\geq2}n^{(k)t}\Big((k\ell-\min\{k,\ell\})(D-2I)-\min\{k,\ell\}I\Big)n^{(\ell)}\end{aligned}$$

Here the second summand $[\ldots]$ appears only if $n^{(1)}\neq 0$. Note that there always is at least one index $k\geq 2$ with $n^{(k)}\neq 2$, since $h\neq 0$. This shows that all summands in the last expression for Δ are non-negative.

The minimal contribution of a non-zero vector $n^{(k)}$, $k\neq 2$, to Δ is

$$k((k-1)a-1)\left(\sum_i n_i^{(k)}\right)^2\geq 2.$$

Thus we always have $\Delta \geq 3$, and even better: $\Delta \geq 4$ in all cases except

$$a = 2, n^{(2)} = (1), n^{(k)} = 0 \text{ for all } k \neq 2.$$

In this case $s = 1$, $n = (2)$, and $d_{11} = 4$, which is the same exceptional case as before.

As in the semisimple case, the union of all sets $G(n)F(n)^g$, g unipotent, is covered by a finite number of such sets. In fact, this is even easier to see: up to conjugation there are only finitely many different nilpotent elements h and hence only finitely many different subschemes $G(n)F(n)^{1+h} \subset Z$.

3.3. *Case: $g \in G(n) \setminus \mathbb{C}^*$ arbitrary.* Consider the multiplicative Jordan decomposition $g = su$, where s is semisimple, u is unipotent and s and u commute. Any endomorphism that commutes with g also commutes with s and u. This implies that $F(n)^g \subset F(n)^s \cap F(n)^u$, so that the general case is covered by 3.1. and 3.2. above. □

3.6. **Return from the normal cone.** Let $v_0 \in H^{\text{even}}(X, \mathbb{Z})$ be a primitive Mukai vector satisfying $(*)$. Let $v = mv_0$ for some multiplicity $m \in \mathbb{N}$. We keep the notation introduced earlier.

Proposition 3.7. — *Let H be an arbitrary ample divisor. Let $E = \bigoplus_{i=1}^s E_i^{\oplus n_i}$ be a polystable sheaf whose stable direct summands E_i satisfy the condition*

$$v(E_i) \in \mathbb{N}v_0 \tag{3.1}$$

Consider a point $[q : \mathcal{H} \to E] \in R^{ss}$ and a slice $S \subset R^{ss}$ to the orbit of $[q]$ as above. Then $\mathcal{O}_{S,[q]}$ is a normal complete intersection domain of dimension

$$\dim \operatorname{Ext}^1(E, E) - \dim \operatorname{Ext}^2(E, E)_0 = 1 + \sum_{i,j} n_i(\dim \operatorname{Ext}^1(E_i, E_j) - \delta_{ij})n_j,$$

that has property R_3 in all cases except the following two:

1. *$s = 1$, $n_1 = 2$, $\dim \operatorname{Ext}^1(E_1, E_1) = 4$,*
2. *$s = 2$, $n_1 = n_2 = 1$, $\dim \operatorname{Ext}^1(E_1, E_2) = 2$.*

Proof. Recall the notation introduced in sections 2.6 and 2.7. By Proposition 3.5, $F = \mu^{-1}(0) = \operatorname{Spec}(\operatorname{gr} A/J)$ is a normal complete intersection variety of dimension

$$\begin{aligned} \dim(F) &= 1 + \sum_{i,j} n_i(\dim \operatorname{Ext}^1(E_i, E_j) - \delta_{ij})n_j \\ &= \dim \operatorname{Ext}^1(E, E) - \dim \operatorname{Ext}^2(E, E)_0. \end{aligned}$$

Therefore, we must have equality at all places in inequality (2.1). Furthermore, since $F = \operatorname{Spec}(\operatorname{gr} A/J)$ is reduced and irreducible, the equality of dimensions implies $J = \operatorname{in}(I)$. It follows that

$$\operatorname{gr}(\widehat{\mathcal{O}}_{S,[q]}) = \operatorname{gr}(A/I) = \operatorname{gr} A/\operatorname{in}(I) = \Gamma(F, \mathcal{O}_F)$$

is a normal complete intersection. In particular, $\operatorname{gr}(\widehat{\mathcal{O}}_{S,[q]})$ is Cohen-Macaulay, hence satisfies S_k for all $k \in \mathbb{N}$. Unless we are in the two exceptional cases, $\operatorname{gr}(\widehat{\mathcal{O}}_{S,[q]})$ is smooth in codimension 3. Now remark that $\mathrm{gr}(\widehat{\mathcal{O}}_{S,[q]}) = \mathrm{gr}(\mathcal{O}_{S,[q]})$ ([1], 10.22) and then use the following proposition which shows that $\mathcal{O}_{S,[q]}$ itself is a normal complete intersection which, unless we are in the two exceptional cases, satisfies R_3. $\square$

Proposition 3.8. — *Let B be a noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field $B/\mathfrak{m} \cong \mathbb{C}$. Let $\mathrm{gr}\, B$ denote the graded ring associated to the $\mathfrak{m}$-adic filtration of B. Then $\dim(B) = \dim(\mathrm{gr}\, B)$, and if $\mathrm{gr}\, B$ is an integral domain or normal or a complete intersection then the same is true for B. Moreover if $\mathrm{gr}\, B$ satisfies R_k and S_{k+1} for some $k \in \mathbb{N}$ then B satisfies R_k.*

Proof. The assertion about integrality and normality is Krull's theorem (see [20] (17.D) Thm 34). The assertions about complete intersections and the property R_k are due to Cavaliere and Niesi ([6], Theorems 3.4 and 3.13). $\square$

Lemma 3.9. — *The assumption* (3.1) *in Proposition 3.7 is satisfied in any of the following two situations:*

1. *H is v–general.*
2. *$E = E_0^{\oplus m}$ for some stable sheaf E_0 with $v(E_0) = v_0$.*

The exceptions of Proposition 3.7 are met in case (B) only, i. e. if $\langle v_0, v_0\rangle = 2$ and $m = 2$.

Proof. Under the assumption that H is v–general one has $v(E_j) = r_j v_0$ for some $r_j \in \mathbb{N}$ and all direct summands E_j of E. Then $\dim \mathrm{Ext}^1(E_i, E_j) = r_i r_j \langle v_0, v_0\rangle \geq 2$. Thus Proposition 3.7 applies. $\square$

Proposition 3.10. — *1. Let H be a v–general ample divisor. Then R^{ss} is normal and locally a complete intersection of dimension $\langle v, v\rangle + 1 + N^2$. In case (C) it has property R_3 and hence is locally factorial.*

2. Suppose that $E = E_0^{\oplus m}$ for some stable sheaf E_0 with $v(E_0) = v_0$. Let H be an arbitrary ample divisor. In case (C), there is an open neighbourhood U of $[E] \in M_v$ such that $\pi^{-1}(U) \subset R^{ss}$ is locally factorial of dimension $\langle v, v\rangle + 1 + N^2$.

Proof. 1. Let $[q] \in R^{ss}$ be a point with closed orbit, and let $S \subset R^{ss}$ be a $\mathrm{PAut}(E)$–equivariant subscheme as in subsection 2.5. By Lemma 3.9 and Proposition 3.7, the local ring $\mathcal{O}_{S,[q]}$ is a normal complete intersection that has property R_3 in case (C). But being normal or locally a complete intersection or having property R_k are open properties [EGA IV 19.3.3, 6.12.9]. Hence there is an open neighbourhood U of $[q]$ in S that is normal, locally a complete intersection, and has property R_3 in case (C). The natural morphism $\mathrm{PGl}(N) \times S \to R^{ss}$ is smooth. Therefore every closed orbit in R^{ss} has an open neighbourhood that is normal, locally a complete intersection, and has property R_3 in case (C). Finally, every $\mathrm{PGl}(N)$–orbit of R^{ss} meets such an open neighbourhood. It follows that R^{ss} is normal, locally a complete intersection. In case (C), R^{ss} is regular in codimension 3 and hence locally factorial due to the following theorem of Grothendieck.

2. The second assertion follows analogously. $\square$

Theorem 3.11. (Grothendieck [12] Exp. XI Cor. 3.14) — *Let B be noetherian local ring. If B is a complete intersection and regular in codimension ≤ 3, then B is factorial.*

4. A basic irreducibility result

The following theorem generalises a beautiful result of Mukai [21].

Theorem 4.1. — *Let X be a projective K3 or abelian surface with an ample divisor H. Let M_v be the moduli space of semistable sheaves associated to a vector $v \in H^{\mathrm{even}}(X,\mathbb{Z})$. Suppose that $Y \subset M_v$ is a connected component parametrising stable sheaves only. Then $M_v = Y$.*

Proof. 1. Since all points in Y correspond to stable sheaves, Y is smooth of expected dimension $\dim(Y) = 2 + \langle v,v\rangle$. Fix a point $[F] \in Y$ and suppose that there is a point $[G] \in M_v \setminus Y$. We shall exploit a beautiful old idea of Mukai [21]: assume for a moment that there were a universal family $\mathbb{F} \in \mathrm{Coh}(Y \times X)$. Let $p: Y \times X \to Y$ and $q: Y \times X \to X$ be the projections. We may then compare the relative Ext-sheaves $\mathrm{Ext}_p^\bullet(q^*F,\mathbb{F})$ and $\mathrm{Ext}_p^\bullet(q^*G,\mathbb{F})$. Since F and G are numerically equal on X, the same is true for the classes of the Ext-sheaves according to the Grothendieck-Riemann-Roch theorem. This will lead to a contradiction.

2. In general, there is no universal family, but the following construction will be sufficient:

Lemma 4.2. — *There is a smooth projective variety Y' that parametrises a family $\mathbb{F}$ of stable sheaves on X with Mukai vector v such that the classifying morphism $f: Y' \to Y$ is surjective, generically finite, and étale over a neighbourhood of $[F]$.*

Proof. Let $R' := Y \times_{M_v} R^{ss}$. Then $R' \to Y$ is a $\mathrm{PGl}(N)$–principal fibre bundle, locally trivial in the étale topology. Moreover, there is a universal epimorphism $\mathcal{O}_{R'} \boxtimes \mathcal{H} \to \mathbb{F}'$. We form the quotient $P := (\mathbb{P}^{N-1} \times R')/\!\!/\,\mathrm{Gl}(N)$. Then P is a smooth projective variety, and the natural morphism $P \to Y$ is locally a product in the étale topology with fibres isomorphic to $\mathbb{P}^{N-1}$. The center $\mathbb{C}^* \subset \mathrm{Gl}(N)$ acts trivially on the family $\mathcal{O}_{\mathbb{P}^{N-1}}(-1) \boxtimes \mathbb{F}'$. Therefore, this sheaf descends to a family $\mathbb{F}_P$ on $P \times X$. Let L be a very ample line bundle on P. Choose a linear subspace $Z \subset \mathbb{P}(H^0(P,L))$ of codimension $N-1$ in such a way that $Y' := Z \cap P$ is smooth and $f: Y' \to Y$ is étale over a neighbourhood of $[F]$. Finally, let $\mathbb{F} := \mathbb{F}_P|_{Y' \times X}$. □

3. Let $f: Y' \to Y$ and $\mathbb{F}$ be chosen as in the lemma and let $p: Y' \times X \to Y'$ and $q: Y' \times X \to X$ denote the two projections. Moreover, let $f^{-1}([F]) = \{p_1, \ldots, p_n\}$.

As G represents a point in $M \setminus Y$ and hence is not isomorphic to any of the stable sheaves E, $[E] \in Y$, one has $\mathrm{Hom}(G,E) = 0 = \mathrm{Ext}^2(G,E)$ for all $[E] \in Y$. It follows that $\mathrm{Ext}_p^0(q^*G,\mathbb{F})$ and $\mathrm{Ext}_p^2(q^*G,\mathbb{F})$ vanish and that $W := \mathrm{Ext}_p^1(q^*G,\mathbb{F})$ is a locally free sheaf on Y' of rank $\langle v,v\rangle = \dim(Y) - 2$.

If G is replaced by F the situation gets more complicated as the dimension of the Ext-groups jumps on the fibre T. There is a complex of locally free $\mathcal{O}_{Y'}$–sheaves

$$0 \longrightarrow A^0 \xrightarrow{\alpha} A^1 \xrightarrow{\beta} A^2 \longrightarrow 0 \tag{4.1}$$

with the property that $\mathrm{Ext}^i_{p_S}(t_X^* q^* G, t_X^* \mathbb{F}) \cong h^i(t^*(A^\bullet))$ for every base change

$$\begin{array}{ccccc} S \times X & \xrightarrow{t_X} & Y' \times X & \xrightarrow{q} & X \\ {\scriptstyle p_S}\downarrow & & {\scriptstyle p}\downarrow & & \\ S & \xrightarrow{t} & Y'. & & \end{array} \tag{4.2}$$

Lemma 4.3. — *The degeneracy locus of α and β is the union of the reduced points $p_1, \ldots, p_n$. Moreover, $\mathrm{rk}\,\alpha(p_i) = \mathrm{rk}\,A^0 - 1$ and $\mathrm{rk}\,\beta(p_i) = \mathrm{rk}\,A^2 - 1$ for $i = 1, \ldots, n$.*

Proof. For all $[E] \in Y$, $E \not\cong F$, one has $\mathrm{Hom}(F, E) = 0 = \mathrm{Ext}^2(F, E)$. This implies that α and β have maximal rank on $Y' \setminus \{p_1, \ldots, p_n\}$. Moreover, $\mathrm{Hom}(F, F) = \mathbb{C} = \mathrm{Ext}^2(F, F)$, and this gives the second assertion of the lemma. It remains to show that the degeneracy locus is reduced. Recall that tangent vectors in $T_{[F]}Y$ correspond bijectively to elements $\gamma \in \mathrm{Ext}^1(F, F)$. Let F_γ be the infinitesimal extension of F over Spec $\mathbb{C}[\varepsilon]$ corresponding to γ. The extension

$$0 \longrightarrow F \xrightarrow{\varepsilon} F_\gamma \longrightarrow F \longrightarrow 0 \tag{4.3}$$

induces a long exact sequence

$$\longrightarrow \mathrm{Ext}^i(F, F) \longrightarrow \mathrm{Ext}^i_{\mathrm{Spec}\ \mathbb{C}[\varepsilon]}(F \otimes \mathbb{C}[\varepsilon], F_\gamma) \longrightarrow \mathrm{Ext}^i(F, F) \xrightarrow{\partial} \mathrm{Ext}^{i+1}(F, F) \longrightarrow$$

where the boundary operator is given by $\partial(e) = \gamma \cup e$. Now $\gamma \cup - : \mathbb{C} = \mathrm{End}(F, F) \to \mathrm{Ext}^1(F, F)$ is clearly injective, and $\gamma \cup - : \mathrm{Ext}^1(F, F) \to \mathrm{Ext}^2(F, F)$ is surjective since the symplectic form on $\mathrm{Ext}^1(F, F)$ is non-degenerate. It follows that $\mathrm{Ext}^0(F, F) \cong \mathrm{Ext}^0_{\mathrm{Spec}\ \mathbb{C}[\varepsilon]}(F \otimes \mathbb{C}[\varepsilon], F_\gamma)$ and $\mathrm{Ext}^2_{\mathrm{Spec}\ \mathbb{C}[\varepsilon]}(F \otimes \mathbb{C}[\varepsilon], F_\gamma) \cong \mathrm{Ext}^2(F, F)$. If the degeneracy locus of α resp. β were not reduced, the corresponding Ext groups should be bigger than $\mathbb{C}$ for at least one γ. The calculation shows that this is not the case. □

4. Let $\sigma : Z \to Y$ denote the blow-up of Y in $[F]$ with exceptional divisor D and similarly $\varphi : Z' \to Y'$ the blow-up of Y' in all points p_i with corresponding exceptional divisors D_i.

$$\begin{array}{ccccccc} D & \subset & Z & \xleftarrow{g} & Z' & \supset & D_i \\ \downarrow & & {\scriptstyle \sigma}\downarrow & & {\scriptstyle \varphi}\downarrow & & \downarrow \\ [F] & \in & Y & \xleftarrow{f} & Y' & \ni & p_i \end{array} \tag{4.4}$$

According to the lemma, the degeneracy locus of both $\varphi^*(\alpha)$ and $\varphi^*(\beta)$ is precisely the smooth divisor $D' = D_1 \cup \ldots \cup D_n$. Therefore these maps factor as follows:

$$\varphi^* A^0 \subset A'^0 \xrightarrow{\alpha'} \varphi^* A^1 \xrightarrow{\beta'} A'^2 \subset \varphi^* A^2, \tag{4.5}$$

where A'^0 and A'^2 are locally free, α' and β' are homomorphisms of maximal rank. Moreover, the line bundles $L := \varphi^*A^2/A'^2$ and $M := A'^0/\varphi^*A^0$ on D' are characterised by the canonical isomorphisms

$$L \otimes \mathcal{O}_{D'} \cong \operatorname{Ext}^2_{D'}(q^*F, \mathcal{O}_D \boxtimes \mathbb{F}|_{f^{-1}([F])\times X}) \cong \operatorname{Ext}^2(F,F) \otimes_{\mathbb{C}} \mathcal{O}_{D'}$$

and

$$\operatorname{Tor}_1^{\mathcal{O}_{Z'}}(M, \mathcal{O}_{D'}) \cong \operatorname{Ext}^0_{D'}(q^*F, \mathcal{O}_D \boxtimes \mathbb{F}|_{f^{-1}([F])\times X}) \cong \operatorname{Hom}(F,F) \otimes_{\mathbb{C}} \mathcal{O}_{D'},$$

implying

$$L \cong \bigoplus_{i=1}^n \mathcal{O}_{D_i} \quad \text{and} \quad M \cong \bigoplus_{i=1}^n \mathcal{O}_{D_i}(D_i). \tag{4.6}$$

5. Let W' denote the middle cohomology of the complex

$$0 \longrightarrow A'^0 \xrightarrow{\alpha'} \varphi^*A^1 \xrightarrow{\beta'} A'^2 \longrightarrow 0.$$

W' is locally free of rank $\dim Y - 2$. We obtain the following equation of Chern classes in $H^*(Z', \mathbb{Z})$:

$$\varphi^* c(A^1 - A^0 - A^2) = c(W' + M - L). \tag{4.7}$$

On the other hand, as $c(F) = c(G)$ in $H^*(X, \mathbb{Z})$, the Grothendieck-Riemann-Roch Theorem yields the following identity in $H^*(Y', \mathbb{Z})$:

$$c(A^1 - A^0 - A^2) = c(\operatorname{Ext}^\bullet_p(q^*F, \mathbb{F})) = c(\operatorname{Ext}^\bullet_p(q^*G, \mathbb{F})) = c(W). \tag{4.8}$$

Combining (4.7) and (4.8), we conclude that

$$c(W') = \varphi^* c(W) \cdot c(L - M) \in H^*(Z', \mathbb{Z}) \tag{4.9}$$

Moreover,

$$c(L-M) = \prod_{i=1}^n \frac{c(\mathcal{O}_{D_i})}{c(\mathcal{O}_{D_i}(D_i))} = \prod_{i=1}^n \frac{1}{c(\mathcal{O}_{Z'}(-D_i))c(\mathcal{O}_{Z'}(D_i))} = 1 + \sum_{k=1}^\infty \sum_{i=1}^n D_i^{2k}.$$

The product of any cohomology class in $H^*(Y', \mathbb{Z})$ of positive degree with any of the classes D_i is zero. It follows that

$$c_{2k}(W') = \varphi^* c_{2k}(W) + \sum_{i=1}^n D_i^{2k} \quad \text{for all } k > 0.$$

The key point now is that both W and W' are vector bundles of rank $\dim(Y) - 2$, so that the Chern classes $c_{\dim(Y)}(W)$ and $c_{\dim(Y)}(W')$ vanish (*cf.* [19], Lemma 4). We get the contradiction

$$0 = \sum_{i=1}^n D_i^{\dim(Y)} = -n.$$

This finishes the proof of Theorem 4.1. □

Theorem 4.4. — *Let v_0 be a primitive Mukai vector satisfying condition $(*)$ and $(**)$. Let $v = mv_0$ and let H be a v–general ample divisor. Then M_v is a normal irreducible variety of dimension $2 + \langle v, v\rangle$.*

This theorem is due to Yoshioka [28] in the case of torsion free sheaves. Using the local information obtained in Proposition 3.10, the basic irreducibility result of Theorem 4.1, we can give a simple direct proof.

Proof. By Proposition 3.10, R^{ss} is normal. As a GIT-quotient of a normal scheme, M_v is also normal. If $m = 1$, all points in $M_v = M_{v_0}$ correspond to stable sheaves and hence M_v is smooth. By Theorem 4.1, M_{v_0} is irreducible. By $(**)$, M_{v_0} is non-empty.

Assume now that $m \geq 2$ and that the assertion of the theorem has been proved for all moduli spaces $M_{m'v_0}$, $1 \leq m' < m$. For any decomposition $m = m' + m''$ with $1 \leq m' \leq m''$, consider the morphism

$$\varphi(m', m'') : M_{m'v_0} \times M_{m''v_0} \longrightarrow M_{mv_0}, \quad ([E'], [E'']) \mapsto [E' \oplus E''],$$

and let $Y(m', m'') \subset M_v$ denote its image. The subschemes $Y(m', m'')$, $1 \leq m' \leq m''$, are the irreducible components of the strictly semistable locus of M_v. Since all $Y(m', m'')$ are irreducible by induction and intersect in the points of the form $[E_0^{\oplus m}]$, $[E_0] \in M_{v_0}$, the strictly semistable locus is connected. Since M_v is normal, the connected components are irreducible. In particular, there is exactly one component that meets the strictly semistable locus. Theorem 4.1 excludes the possibility of a component that does not meet the strictly semistable locus. □

5. Factoriality of moduli spaces

Proposition 5.1. — *Let v_0 be a primitive Mukai vector satisfying $(*)$. Let $v = mv_0$ for some $m \in \mathbb{N}_0$. Assume that*

- *either $E = E_0^{\oplus m}$, for some E_0 stable with $v(E_0) = v_0$, and H is arbitrary,*
- *or E is arbitrary polystable with $v(E) = v$, and H is v–general.*

Assume further that case (C) applies. Then M_v is locally factorial at $[E]$ if and only if the isotropy subgroup $\mathrm{PGl}(N)_{[q]} \cong \mathrm{PAut}(E)$ of any point $[q]$ in the closed orbit in $\pi^{-1}([E]) \subset R^{ss}$ acts trivially on the fibre $L([q])$ for every $\mathrm{PGl}(N)$-linearised line bundle L on an invariant open neighbourhood of the orbit of $[q]$.

Proof. This is Drezet's Théorème A [10]. In Drezet's situation the Quot scheme R^{ss} is smooth. However, all his arguments go through under the weaker hypothesis that R^{ss} is locally factorial in a $\mathrm{PGl}(N)$–equivariant open neighbourhood of the closed orbit in the fibre $\pi^{-1}([E])$. But this is true under the given hypothesis due to Proposition 3.10 □

Corollary 5.2. — *Let E_0 be a stable sheaf with Mukai vector $v(E_0) = v_0$ satisfying $(*)$ and assume that $v = mv_0$ satisfies (C). Then M_v is locally factorial at $[E_0^{\oplus m}]$.*

Proof. The isotropy subgroup of any point $[q]$ in the closed orbit in $\pi^{-1}([E_0^{\oplus m}]) \subset R^{ss}$ is isomorphic to $\mathrm{PGl}(m)$ and therefore has no non-trivial characters. Hence the action of $\mathrm{PGl}(m)$ on $L([q])$ is necessarily trivial (notations as in Proposition 5.1). □

Theorem 5.3. — *Let v_0 be a primitive Mukai vector satisfying* $(*)$ *and* $(**)$*. Assume that $v = mv_0$, $m \in \mathbb{N}$, satisfies (C) and let H be a v–general ample divisor. Then M_v is locally factorial.*

Proof. Let $[E] \in M_v$ be an arbitrary point that is represented by the polystable sheaf $E = \bigoplus_{i=1}^s E_i^{\bigoplus n_i}$, and let $[q : \mathcal{H} \to E]$ be a point in the closed orbit in $\pi^{-1}([E]) \subset R^{ss}$. Since H is v–general, the Mukai vectors of the stable direct summands E_i have the form

$$v(E_i) = m_i v_0, \quad m_i \in \mathbb{N}, \quad \sum_{i=1}^s m_i n_i = m.$$

We repeat the construction in section 2.5 for each of the Mukai vectors $m_i v_0$, $i = 1, \ldots, s$. Note that we can choose a sufficiently large integer k that works for all Mukai vectors simultaneously. Let $P_i(z) = -m_i\langle v_0, v(\mathcal{O}_X(-zH)\rangle$, $N_i = P_i(k)$ and $\mathcal{H}_i = \mathcal{O}_X(-kH)^{\oplus N_i}$. Then $N = \sum_i n_i N_i$ and $\mathcal{H} = \bigoplus_i \mathcal{H}_i^{\oplus n_i}$. Moreover there are parameter spaces $R_i^{ss} \subset \mathrm{Quot}_{X,H}(\mathcal{H}_i, P_i)$ with $\mathrm{PGl}(N_i)$-actions and quotient maps $\pi_i : R_i^{ss} \to M_{m_i v_0}$. Finally there is a canonical map

$$\Phi : \prod_i R_i^{ss} \longrightarrow R^{ss}, \quad \big([\mathcal{H}_i \to F_i]\big)_i \mapsto [\mathcal{H} = \bigoplus_i \mathcal{H}_i^{\oplus n_i} \to \bigoplus_i F_i^{\oplus n_i}].$$

Let Z denote the image of Φ. It has the following properties:

- By Theorem 4.4, the moduli spaces $M_{m_i v_0}$ are irreducible. It follows that the schemes R_i^{ss} and Z are irreducible, too.
- Z contains the point $[q]$ and as well a point $[q' : \mathcal{H} \to E_0^{\oplus m}]$ for some stable sheaf E_0 with $v(E_0) = v_0$.
- The group $G := \big(\prod_i \mathrm{Gl}(n_i)\big)/\mathbb{C}^* \subset \mathrm{PGl}(N)$ fixes Z pointwise. It equals the stabiliser subgroup of $[q]$ and is contained in the stabiliser subgroup of $[q']$.

Now let L be a $\mathrm{PGl}(N)$–linearised line bundle on R^{ss}. The group G acts on $L|_Z$ with a locally constant character, which must in fact be constant, since Z is connected. Moreover, the action is trivial at the point $[q']$ according to the proof of Corollary 5.2. Thus the character is trivial everywhere on Z and in particular at $[q]$. According to Drezet's criterion (Proposition 5.1), M_v is locally factorial at $[E]$. □

Remark 5.4. It is also known that the moduli space of semi-stable torsion free sheaves on the projective plane is locally factorial by the work of Drezet [9]. However it may be false for other surfaces as has been observed by Le Potier: the moduli space $M_{\mathbb{P}^1\times\mathbb{P}^1}(2,0,2)$ is not locally factorial at the point represented by $\mathcal{O}(1,-1) \oplus \mathcal{O}(-1,1)$ (see [10], p. 106).

6. Symplectic resolutions

Let v_0 be a primitive Mukai vector satisfying $(*)$ and $(**)$. Let $v = mv_0$ and let H be a v–general divisor. Recall that the following three cases are possible:

(A) $m = 1$.
(B) $m = 2$ and $\langle v_0, v_0\rangle = 2$.

(C) $m \geq 3$, or $m = 2$ and $\langle v_0, v_0 \rangle \geq 4$.

In case (A) the moduli space M_v consists only of stable sheaves. It is irreducible and smooth of dimension $2 + \langle v, v \rangle$. Mukai [21] has defined a symplectic structure on M_v.

Proposition 6.1. — *Assume that $m \geq 2$. The singular locus $M_{v,\mathrm{sing}}$ of M_v is non-empty and equals the semistable locus. The irreducible components of $M_{v,\mathrm{sing}}$ correspond to integers m', $1 \leq m' \leq m/2$, and have codimension $2m'(m-m')\langle v_0, v_0 \rangle - 2$, respectively. In particular,* $\operatorname{codim} M_{v,\mathrm{sing}} = 2$ *in case (B) and ≥ 4 in case (C).*

Proof. Recall the varieties $Y(m', m'')$ introduced in the proof of Theorem 4.4. The union of the $Y(m', m'')$ is the strictly semistable locus. The maps

$$\varphi(m', m'') : M_{m'v_0} \times M_{m''v_0} \to Y(m', m'')$$

are finite and surjective, hence

$$\begin{aligned} \operatorname{codim}(Y(m', m'')) &= 2 + m^2\langle v_0, v_0 \rangle - (2 + m'^2\langle v_0, v_0 \rangle) - (2 + m''^2\langle v_0, v_0 \rangle) \\ &= 2m'm''\langle v_0, v_0 \rangle - 2. \end{aligned}$$

Clearly, the codimension 2 is attained only if $m' = m'' = 1$ and $\langle v_0, v_0 \rangle = 2$, which is case (B). As M_v is smooth in all stable points, it remains to show that the strictly semistable points are really singular. For this it suffices to show that M_v is singular at a generic point $[E = E' \oplus E''] \in Y(m', m'')$, where E' and E'' are stable sheaves with $v(E') = m'v_0$ and $v(E'') = m''v_0$. In this case, $\mathrm{PAut}(E) \cong \mathbb{C}^*$, $\mathrm{Ext}^2(E, E)_0 \cong \mathbb{C}$, and the Kuranishi map $\mathrm{Ext}^2(E, E)_0 \to \mathbb{C}[\mathrm{Ext}^1(E, E)]^\wedge$ is completely described by an invariant function $f \in \mathbb{C}[\mathrm{Ext}^1(E, E)]^\wedge$. It follows, that

$$\widehat{\mathcal{O}}_{M_v,[E]} \cong \left(\mathbb{C}[\mathrm{Ext}^1(E, E)]^\wedge\right)^{\mathbb{C}^*} \Big/ (f).$$

Now $\mathbb{C}^*$ acts on the four summands of

$$\mathrm{Ext}^1(E, E) = \mathrm{Ext}^1(E', E') \oplus \mathrm{Ext}^1(E', E'') \oplus \mathrm{Ext}^1(E'', E') \oplus \mathrm{Ext}^1(E'', E'')$$

with weights 0, 1, -1, and 0. It follows that

$$\mathrm{Ext}^1(E, E)/\!\!/\mathbb{C}^* = \mathrm{Ext}^1(E', E') \times C \times \mathrm{Ext}^1(E'', E''),$$

where $C \subset M(d, \mathbb{C})$ is the cone of matrices of rank ≤ 1 and

$$d = \dim \mathrm{Ext}^1(E', E'') = m'm''\langle v_0, v_0 \rangle \geq 2.$$

Since the quotient of a singular local ring by a non-zero divisor cannot become regular, $\widehat{\mathcal{O}}_{M_v,[E]}$ is singular. □

Theorem 6.2. — *Suppose that v belongs to case (C). Then M_v is a locally factorial symplectic variety of dimension $2 + \langle v, v \rangle$. The singular locus is non-empty and has codimension 4. All singularities are symplectic, but there is no open neighbourhood of a singular point in M_v that admits a projective symplectic resolution.*

Proof. We have already seen that M_v is a locally factorial variety. Mukai [21] constructed a non-degenerate 2-form on M_v^s. This form is closed even if M_v^s is not projective ([13] Prop. 10.3.2). By Flenner's theorem [11] this form extends to any resolution of the singularities of M_v. Hence the singularities are symplectic in the sense of Beauville [5]. Now let $[E] \in M_v$ be a singular point and let $U \subset M_v$ be an open neighbourhood of $[E]$. A projective symplectic resolution of U is a projective resolution $\sigma : U' \to U$ of the singularities of U such that the restriction of the symplectic form on M_v^s to U^{reg} extends to a symplectic form on U'. In such a case the morphism σ would have to be semismall according to a result of Kaledin, [14] Lemma 2.11. As the singular locus of U has codimension ≥ 4 according to Proposition 6.1, the exceptional locus of σ has codimension ≥ 2 in U'. On the other hand $\mathcal{O}_{M_v,[E]}$ is factorial by Theorem 5.3. This implies that the exceptional locus must be a divisor (see [7] no. 1.40 p. 28). $\square$

Remark 6.3. — 1) The completion of a factorial local ring is not factorial in general. The local rings of the moduli spaces of type (C) provide nice examples of this phenomenon. Pushing the arguments in the previous proof a bit further, one sees that

$$\widehat{\mathcal{O}}_{M_v,[E]} \cong \mathbb{C}[\mathrm{Ext}^1(E',E') \oplus \mathrm{Ext}^1(E'',E'')]^\wedge \mathbin{\widehat{\otimes}} B,$$

where B is the completed coordinate ring of the cone $C_0 \subset C \subset M(d,\mathbb{C})$ of traceless matrices of rank ≤ 1, with $d \geq 4$. But $\widehat{\mathcal{O}}_{M_v,[E]}$ cannot be factorial: the vertex of C_0 is an isolated singularity of codimension ≥ 6, and there are two small symplectic resolutions $T^*\mathbb{P}(\mathrm{Ext}^1(E',E'')) \to C_0 \leftarrow T^*\mathbb{P}(\mathrm{Ext}^1(E'',E'))$. We see that in this case $\mathcal{O}_{M,[E]}$ is factorial due to Theorem 5.3, but $\widehat{\mathcal{O}}_{M,[E]}$ is not. Geometrically, what happens is this: an irreducible Weil divisor becomes reducible after completion; while the whole thing still is a Cartier divisor, some of its newly acquired irreducible components need not be.

2) On the other hand, for polystable sheaves $E_0^{\oplus m}$ with E_0 stable and $v(E_0)$ satisfying $(*)$, the completed local ring $\widehat{\mathcal{O}}_{M_v,[E_0^{\oplus m}]}$ is factorial. In fact, the proof of proposition 3.7 shows that $\widehat{\mathcal{O}}_{S,[q]}$ is factorial. Moreover, the stabiliser is isomorphic to $\mathrm{PGl}(m)$ hence has no non-trivial characters. Under these conditions one can show that the invariant ring $(\widehat{\mathcal{O}}_{S,[q]})^{\mathrm{PGl}(m)} \simeq \widehat{\mathcal{O}}_{M_v,[E_0^{\oplus m}]}$ is also factorial.

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